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Hermite-Hadamard Type Inequalities Involving Nonlocal Conformable Fractional Integrals

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ABSTRACT

Since the so-called Hermite-Hadamard inequality for a convex function was presented, its extensions, refinements, and variants, which are called Hermite-Hadamard type inequalities, have been extensively investigated. In this paper, we aim to establish two Hermite-Hadamard type inequalities and an identity for convex functions associated with known fractional conformable integral operators. Also the results presented here are indicated to reduce to relatively simple known results.

Keywords: Beta function, fractional conformable integral operators, Hermite-Hadamard type inequalities, incomplete beta function, Riemann-Liouville fractional integrals.

1. Introduction and Preliminaries

Let I be an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ be a convex function. Then the following chain of inequalities holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \quad (a,\, b \in I,\, a < b), \tag{1}$$

which is called Hermite-Hadamard inequality. Here and in the sequel, we denote $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, and $\mathbb{Z}_0^-$ by the sets of complex numbers, real numbers, positive real numbers, and non-positive integers, respectively, and let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. A variety of extensions, refinements, and variants of the Hermite-Hadamard inequality (1) along with new proofs have been broadly investigated (see, e.g., Abdeljawad et al. (2020), Baleanu et al. (2020), Chu et al. (2016), Dragomir and Pearce (2000), Han et al. (2020), Iqbal et al. (2020), Khan et al. (2017), Khan and Khan (2021), Khan et al. (2020), Mitrinović and Lacković (1985), Mohammed and Abdeljawad (2020), Mohammed and Hamasalh (2019), Mohammed and Brevik (2020), Ozdemir et al. (2013), Set et al. (2010), and the references cited therein). In this paper, we aim to establish two Hermite-Hadamard type inequalities and an identity for convex functions involving the fractional conformable integral operators (7) and (8). Also we point out that our main results reduce to some relatively simple known results.

To do this, we recall some definitions and known results. Let $[a, b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$ and $f \in L_1[a, b]$.

The *left-sided* and *right-sided Riemann-Liouville fractional integrals* $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha \in \mathbb{C}$ $(Re(\alpha) > 0)$ are defined, respectively, by

$$\left(J^{\alpha}_{a+}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)\,dt \quad (x > a;\ Re(\alpha) > 0), \tag{2}$$

and

$$\left(J^{\alpha}_{b-}f\right)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)\,dt \quad (x < b;\ Re(\alpha) > 0). \tag{3}$$

Here $\Gamma(\alpha)$ is the familiar Gamma function (see, e.g., (Srivastava and Choi, 2012, Section 1.1)). For more details and properties concerning the fractional integral operators (2) and (3), we refer the reader, for example, to the works Dahmani (2010), Dahmani et al. (2010), Gorenflo and Mainardi (1997), Kilbas et al. (2006), Samko et al. (1993), Sarikaya et al. (2013), Set et al. (2015, 2014) and the references therein.

Sarikaya et al. (2013) established a Hermite-Hadamard type integral inequality involving Riemann-Liouville fractional integrals as in the following theorem.

Theorem A. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. Also let f be a convex function on $[a,b]$ and $\alpha \in \mathbb{R}^+$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

The case $\alpha = 1$ of (4) reduces to the Hermite-Hadamard inequality (1). Also we recall two more results in Lemma A and Theorem B.

Lemma A. (Sarikaya et al. (2013)) *Let a mapping $f : [a,b] \to \mathbb{R}$ $(a < b)$* be such that *$f' \in L[a,b]$ and $\alpha \in \mathbb{R}^+$. Then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b_-}^\alpha f(a)\right] \\ &= \frac{b-a}{2}\int_0^1 \left[(1-t)^\alpha - t^\alpha\right] f'(ta+(1-t)b)\,dt. \end{aligned} \tag{5}$$

Theorem B. (Sarikaya et al. (2013)) *Let a mapping $f : [a,b] \to \mathbb{R}$ $(a < b)$* be such that *$f' \in L[a,b]$ and $\alpha \in \mathbb{R}^+$. Then*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + I_{b_-}^\alpha f(a)\right]\right| \\ &\leq \frac{b-a}{2(\alpha+1)}\left(1 - \frac{1}{2^\alpha}\right)\left(|f'(a)| + |f'(b)|\right). \end{aligned} \tag{6}$$

Jarad et al. (2017) introduced the left- and right-fractional conformable integral operators defined $(Re(\beta) > 0)$, respectively, by

$${}_a^\beta\mathcal{J}^\alpha f(x) = \frac{1}{\Gamma(\beta)}\int_a^x \left(\frac{(x-a)^\alpha - (t-a)^\alpha}{\alpha}\right)^{\beta-1} \frac{f(t)}{(t-a)^{1-\alpha}}dt, \tag{7}$$

and

$${}^\beta\mathcal{J}_b^\alpha f(x) = \frac{1}{\Gamma(\beta)}\int_x^b \left(\frac{(b-x)^\alpha - (b-t)^\alpha}{\alpha}\right)^{\beta-1} \frac{f(t)}{(b-t)^{1-\alpha}}dt. \tag{8}$$

For more detailed properties and certain special cases of the integral operators (7) and (8), we refer to Jarad et al. (2017).

2. Hermite-Hadamard Type Inequalities for the Nonlocal Conformable Fractional Integrals

We establish Hermite-Hadamard type inequalities involving the fractional conformable integral operators (7) and (8) in the following theorem.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ $(a < b)$ and $f \in L[a,b]$. Also let $\alpha, \beta \in \mathbb{R}^+$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta}{2(b-a)^{\alpha\beta}} \left[{}_a^\beta\mathcal{J}^\alpha f(b) + {}^\beta\mathcal{J}_b^\alpha f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{9}$$

Proof. Since f is a convex function on $[a,b]$, we have

$$f\left(\frac{x+y}{2}\right) \le \frac{1}{2}f(x) + \frac{1}{2}f(y) \quad (x,\, y \in [a,b]). \tag{10}$$

Setting $x = ta + (1-t)b$ and $y = (1-t)a + tb$ in (10), we obtain

$$2f\left(\frac{a+b}{2}\right) \le f(ta+(1-t)b) + f((1-t)a+tb) \quad (0 \le t \le 1). \tag{11}$$

Multiplying both sides of the inequality (11) by $\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta-1} t^{\alpha-1}$ and integrating each side of the resulting inequality with respect to t on $[0,1]$, we get

$$\begin{aligned}
&2f\left(\frac{a+b}{2}\right)\int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta-1} t^{\alpha-1}dt \\
\le\ & \int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta-1} t^{\alpha-1} f(ta+(1-t)b)dt \\
&\quad + \int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta-1} t^{\alpha-1} f((1-t)a+tb)dt \\
=\ & \frac{1}{b-a}\int_a^b \left(\frac{1-\left(\frac{b-u}{b-a}\right)^\alpha}{\alpha}\right)^{\beta-1} \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)du \\
& + \frac{1}{b-a}\int_a^b \left(\frac{1-\left(\frac{v-a}{b-a}\right)^\alpha}{\alpha}\right)^{\beta-1} \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)dv \\
=\ & \frac{1}{(b-a)^{\alpha\beta}}\int_a^b \left(\frac{(b-a)^\alpha-(b-u)^\alpha}{\alpha}\right)^{\beta-1} \frac{f(u)}{(b-u)^{1-\alpha}}du \\
& + \frac{1}{(b-a)^{\alpha\beta}}\int_a^b \left(\frac{(b-a)^\alpha-(v-a)^\alpha}{\alpha}\right)^{\beta-1} \frac{f(v)}{(v-a)^{1-\alpha}}dv \\
=\ & \frac{\Gamma(\beta)}{(b-a)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_b^\alpha f(a) + {}_a^\beta\mathcal{J}^\alpha f(b)\right].
\end{aligned}$$

Noting

$$\int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^{\beta-1} t^{\alpha-1}dt = \frac{1}{\beta\alpha^\beta},$$

we have

$$2f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)\alpha^\beta}{(b-a)^{\alpha\beta}} \left[{}^\beta\mathcal{J}_b^\alpha f(a) +_a^\beta \mathcal{J}^\alpha f(b)\right],$$

which is the first inequality of (9).

For the second inequality of (9), using convexity of f on $[a,b]$, we have, for $0 \le t \le 1$,

$$f(ta+(1-t)b) \le tf(a) + (1-t)f(b),$$

and

$$f(tb+(1-t)a) \le tf(b) + (1-t)f(a).$$

Adding these inequalities side by side, we obtain

$$f(ta+(1-t)b) + f(tb+(1-t)a) \le f(a)+f(b) \quad (0 \le t \le 1). \tag{12}$$

Multiplying both sides of the inequality (12) by $\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta-1} t^{\alpha-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we get the second inequality of (9). □

We present analogs for the fractional conformable integral operators (7) and (8) of the results in Lemma A and Theorem B, which are asserted, respectively, in Lemma 2.1 and Theorem 2.2.

Lemma 2.1. *Let a function $f : [a,b] \to \mathbb{R}$ $(a < b)$ be such that $f' \in L[a,b]$. Also let α, $\beta \in \mathbb{R}^+$. Then*

$$\begin{aligned} &\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)\alpha^\beta}{2(b-a)^{\alpha\beta}} \left[{}^\beta\mathcal{J}_b^\alpha f(a) +_a^\beta \mathcal{J}^\alpha f(b)\right] \\ &= \frac{(b-a)\alpha^\beta}{2} \int_0^1 \left[\left(\frac{1-t^\alpha}{\alpha}\right)^\beta - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta\right] f'(ta+(1-t)b)\,dt. \end{aligned} \tag{13}$$

Proof. Let I be the integral on the right side of (13) and let $I := I_1 - I_2$ where

$$I_1 := \int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^\beta f'(ta+(1-t)b)\,dt,$$

and

$$I_2 := \int_0^1 \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta f'(ta+(1-t)b)\,dt.$$

By using integrating by parts, we have

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{1-t^\alpha}{\alpha}\right)^\beta f'(ta+(1-t)b)dt \\
&= \left(\frac{1-t^\alpha}{\alpha}\right)^\beta \frac{f(ta+(1-t)b)}{a-b}\bigg|_0^1 - \int_0^1 \beta\left(\frac{1-t^\alpha}{\alpha}\right)^{\beta-1} t^{\alpha-1}\frac{f(ta+(1-t)b)}{b-a}dt \\
&= \frac{1}{\alpha^\beta}\frac{f(b)}{b-a} - \frac{\beta}{b-a}\frac{\Gamma(\beta)}{(b-a)^{\alpha\beta}}\,{}^\beta\mathcal{J}_b^\alpha f(a) \\
&= \frac{1}{b-a}\left(\frac{f(b)}{\alpha^\beta} - \frac{\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\,{}^\beta\mathcal{J}_b^\alpha f(a)\right).
\end{aligned} \tag{14}$$

Similarly we have

$$I_2 = -\frac{1}{b-a}\left(\frac{f(a)}{\alpha^\beta} - \frac{\Gamma(\beta+1)}{(b-a)^{\alpha\beta}}\,{}_a^\beta\mathcal{J}^\alpha f(b)\right). \tag{15}$$

Using the results (14) and (15) in $I := I_1 - I_2$ and multiplying both sides of the resulting identity by $\frac{b-a}{2}\alpha^\beta$, we are led to the equality (13).

□

We recall Beta function $B(\alpha,\beta)$ and incomplete Beta function $B_x(\alpha,\beta)$ (see, e.g., (Srivastava and Choi, 2012, Section 1.1))

$$B(\alpha,\,\beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\min\{Re(\alpha),\,Re(\beta)\} > 0), \\ \displaystyle\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\,\beta \in \mathbb{C}\setminus\mathbb{Z}_0^-\right) \end{cases} \tag{16}$$

and

$$B_x(\alpha,\,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad (Re(\alpha) > 0). \tag{17}$$

Theorem 2.2. *Let a function $f : [a,b] \to \mathbb{R}$ $(a < b)$ be such that $f' \in L[a,b]$. Also let $|f'|$ be a convex function on $[a,b]$ and $\alpha,\,\beta \in \mathbb{R}^+$. Then*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)\alpha^\beta}{2(b-a)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_b^\alpha f(a) + {}_a^\beta\mathcal{J}^\alpha f(b)\right]\right| \\
&\quad \le \frac{b-a}{2\alpha}\left\{2B_{\left(\frac{1}{2}\right)^\alpha}\left(\frac{1}{\alpha},\beta+1\right) - B\left(\frac{1}{\alpha},\beta+1\right)\right\}\left(|f'(a)|+|f'(b)|\right).
\end{aligned} \tag{18}$$

Proof. Let L be the left member of the inequality (18). Using Lemma 2.1 and convexity of $|f'|$, we get

$$\begin{aligned} L \leq & \frac{(b-a)\alpha^{\beta}}{2} \int_0^1 \left| \left[\left(\frac{1-t^{\alpha}}{\alpha} \right)^{\beta} - \left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\beta} \right] \right| \big(t|f'(a)| + (1-t)|f'(b)| \big)\, dt \\ \leq & \frac{(b-a)\alpha^{\beta}}{2} \left\{ \int_0^{\frac{1}{2}} \left| \left[\left(\frac{1-t^{\alpha}}{\alpha} \right)^{\beta} - \left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\beta} \right] \right| \big(t|f'(a)| + (1-t)|f'(b)| \big)\, dt \right. \\ & \left. + \int_{\frac{1}{2}}^1 \left| \left[\left(\frac{1-(1-t)^{\alpha}}{\alpha} \right)^{\beta} - \left(\frac{1-t^{\alpha}}{\alpha} \right)^{\beta} \right] \right| \big(t|f'(a)| + (1-t)|f'(b)| \big)\, dt \right\}. \end{aligned} \tag{19}$$

Using the following inequalities

$$t^{\alpha} \leq (1-t)^{\alpha} \;\; (0 \leq t \leq 1/2), \quad \text{and} \quad t^{\alpha} \geq (1-t)^{\alpha} \;\; (1/2 \leq t \leq 1)$$

in the last expression of (19), we obtain

$$\begin{aligned} L \leq & \frac{b-a}{2} \left\{ |f'(a)| \int_0^{\frac{1}{2}} \left[t\,(1-t^{\alpha})^{\beta} - t\,\{1-(1-t)^{\alpha}\}^{\beta} \right] dt \right. \\ & + |f'(b)| \int_0^{\frac{1}{2}} \left[(1-t)\,(1-t^{\alpha})^{\beta} - (1-t)\,\{1-(1-t)^{\alpha}\}^{\beta} \right] dt \\ & + |f'(a)| \int_{\frac{1}{2}}^1 \left[t\,\{1-(1-t)^{\alpha}\}^{\beta} - t\,(1-t^{\alpha})^{\beta} \right] dt \\ & \left. + |f'(b)| \int_{\frac{1}{2}}^1 \left[(1-t)\,\{1-(1-t)^{\alpha}\}^{\beta} - (1-t)\,(1-t^{\alpha})^{\beta} \right] dt \right\}. \end{aligned} \tag{20}$$

In view of (16) and (17), we find

$$\int_0^x t^{\gamma}\,(1-t^{\alpha})^{\beta}\, dt = \frac{1}{\alpha} B_{x^{\alpha}} \left(\frac{\gamma+1}{\alpha}, \beta+1 \right) \tag{21}$$

$$\left(\alpha,\, \beta \in \mathbb{R}^+;\, \gamma \in \mathbb{R}_0^+;\, 0 \leq x \leq 1 \right);$$

$$\begin{aligned} \int_0^x t\,\{1-(1-t)^{\alpha}\}^{\beta}\, dt = & \frac{1}{\alpha} \left[B\left(\frac{1}{\alpha}, \beta+1 \right) - B\left(\frac{2}{\alpha}, \beta+1 \right) \right. \\ & \left. + B_{x^{\alpha}}\left(\frac{2}{\alpha}, \beta+1 \right) - B_{x^{\alpha}}\left(\frac{1}{\alpha}, \beta+1 \right) \right] \end{aligned} \tag{22}$$

$$\left(\alpha,\, \beta \in \mathbb{R}^+;\, 0 \leq x \leq 1 \right);$$

$$\begin{aligned}&\int_0^x (1-t)^\gamma \; \{1-(1-t)^\alpha\}^\beta \; dt \\ &\qquad = \frac{1}{\alpha}\left[B\left(\frac{\gamma+1}{\alpha}, \beta+1\right) - B_{(1-x)^\alpha}\left(\frac{\gamma+1}{\alpha}, \beta+1\right)\right] \\ &\qquad\qquad \left(\alpha,\, \beta \in \mathbb{R}^+;\; \gamma \in \mathbb{R}_0^+;\; 0 \le x \le 1\right).\end{aligned} \tag{23}$$

Using (21), (22) and (23) to evaluate the integrals in (20), we obtain the desired inequality (18).

□

Theorem 2.3. *Let a function* $f : [a,b] \to \mathbb{R}$ $(a < b)$ *be such that* $f' \in L[a,b]$. *Also let* $|f'|$ *be a convex function on* $[a,b]$ *and* $\alpha,\, \beta \in \mathbb{R}^+$. *Further let* $p > 1$ *and* $M_{[a,b]}\left(|f'|\right) := \max\left\{|f'(a)|,\, |f'(b)|\right\}$. *Then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)\alpha^\beta}{2(b-a)^{\alpha\beta}}\left[{}^\beta\mathcal{J}_b^\alpha f(a) + {}_a^\beta\,\mathcal{J}^\alpha f(b)\right]\right| \\ &\qquad \le \frac{b-a}{\alpha^{\frac{1}{p}}}\, M_{[a,b]}\left(|f'|\right)\left\{B\left(\frac{1}{\alpha}, p\beta+1\right)\right\}^{\frac{1}{p}}.\end{aligned} \tag{24}$$

Proof. Let D be the left-hand side of the inequality (24). Using (13), we obtain

$$\begin{aligned}D &\le \frac{(b-a)\alpha^\beta}{2}\int_0^1 \left|\left(\frac{1-t^\alpha}{\alpha}\right)^\beta - \left(\frac{1-(1-t)^\alpha}{\alpha}\right)^\beta\right| |f'(ta+(1-t)b)| \; dt \\ &\le (b-a)\int_0^1 (1-t^\alpha)^\beta \; |f'(ta+(1-t)b)| \; dt.\end{aligned}$$

Here, let $\frac{1}{p} + \frac{1}{q} = 1$ $(p > 1)$. Then employing Hölder's inequality (see, e.g., (Royden, 1968, p. 113)) in the last expression of the above inequality, we get

$$D \le (b-a)\left(\int_0^1 (1-t^\alpha)^{p\beta} \; dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(|f'(ta+(1-t)b)|\right)^q \; dt\right)^{\frac{1}{q}}. \tag{25}$$

We have

$$\int_0^1 (1-t^\alpha)^{p\beta} \; dt = \frac{1}{\alpha}\int_0^1 (1-u)^{p\beta} \; u^{\frac{1}{\alpha}-1} du = \frac{1}{\alpha} B\left(\frac{1}{\alpha}, p\beta+1\right). \tag{26}$$

Since $|f'|$ is a convex function on $[a,b]$, we find

$$|f'(ta+(1-t)b)| \le t\; |f'(a)| + (1-t)\,|f'(b)| \le M_{[a,b]}\left(|f'|\right). \tag{27}$$

Finally, using the identity (26) and the inequality (27) in (25), we derive the inequality (24). □

3. Concluding Remark

The results presented here, being general, can be reduced to yield many relatively simple inequalities and identities for convex functions associated with certain fractional integral operators. For example, the case $\alpha = 1$ in the inequality (9), the result (13), and the inequality (18) are easily seen to reduce to the inequality (4), the equality (5), and the inequality (6), respectively.

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